Enumerating Optimal Cost-Constrained Adjustment Sets

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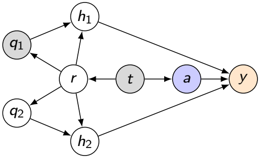
Measuring the Causal Effect

Observational Dist. $+$ Causal Graph $\Rightarrow$ Interventional Dist.

$P(q_2, h_1, r, h_2, a, y)$ $P(Y \mid do(a))$



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From $P(Y \mid do(a))$ we can compute quantities of interest:

ATE: $\mathbb{E}_P(Y \mid do(a)) - \mathbb{E}_P(Y \mid do(a'))$

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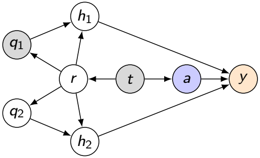

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Yes. $P(y \mid do(a)) = \sum_r P(y \mid a, r) \cdot P(r)$

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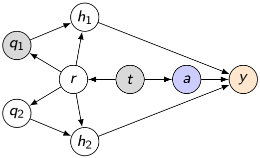

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Can the causal effect of a on y be identified from the observational dist. ?

$$\text{Yes.} \; P(y \mid do(a)) = \sum_r P(y \mid a, r) \cdot P(r)$$

In this example, $\{r\}$ is an **adjustment set** for estimating the causal effect of a on y. Since r is observed, then $\{r\}$ is a **valid adjustment set**, and hence the causal effect of a on y is **identifiable**.

[1] Illustration based on Rotnitzky 2021.

Adjustment Sets

Definition (Adjustment Sets)

Given a DAG G, and pairwise disjoint $X, Y, Z \subseteq V(G)$, Z is an *adjustment set* for estimating the causal effect of X on Y if, for every distribution P that factorizes according to G:

$$P(\boldsymbol{Y} \mid do(\boldsymbol{X})) = \begin{cases} P(\boldsymbol{Y} \mid \boldsymbol{X}) & \text{if } Z = \emptyset \\ \sum_{\boldsymbol{Z}} P(\boldsymbol{Y} \mid \boldsymbol{X}, \boldsymbol{Z}) P(\boldsymbol{Z}) & \text{otherwise} \end{cases}$$

The choice of adjustment set Z affects:

1. Time to compute interventional dist. $P(\boldsymbol{Y} \mid do(\boldsymbol{X}))$ and the derived quantities of interest. Takes time exponential in $|\boldsymbol{Z}|$.
2. The variance $\sigma^2_{\boldsymbol{Z}}(P)$ of the (asymptotically) normally-distributed estimator $\hat{\mathbb{E}}_P(\boldsymbol{Y} \mid do(\boldsymbol{X}))$ Smucler et al. 2021, Smucler and Rotnitzky 2022, Runge 2021.

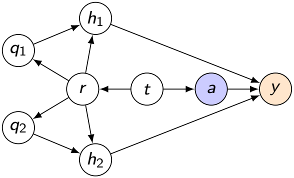
Natural Requirements from Adjustment Sets

Let G be a DAG with observable variables $R \subseteq V(G)$. We wish to find adjustment sets $Z \subseteq V(G)$ that are:

1. Valid : $Z \subseteq R$.
2. Non-Redundant (Minimal) : there is no $Z' \subsetneq Z$ that is an adjustment set for estimating the causal effect.
3. Small size (or weight); of size at most k.
4. Yields estimator with low variance $\sigma^2_{\boldsymbol{Z}}(P)$:

$$\hat{\mathbb{E}}_P(\boldsymbol{Y} \mid do(\boldsymbol{X})) = \sum_{\boldsymbol{Z}} \hat{\mathbb{E}}_P(\boldsymbol{Y} \mid do(\boldsymbol{X}), \boldsymbol{Z}) \cdot P(\boldsymbol{Z}).$$

Examples of Adjustment Sets



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Minimal adjustment sets for estimating the causal effect of a on y: $\{t\}$, $\{r\}$, $\{h_1, q_2\}$, $\{h_2, q_1\}$, $\{h_1, h_2\}$, $\{q_1, q_2\}$.

- Adjustment set(s) that optimize for size: $\{t\}$, $\{r\}$.
- Adjustment set(s) that optimize for variance: $\{h_1, h_2\}$ Smucler et al. 2021.

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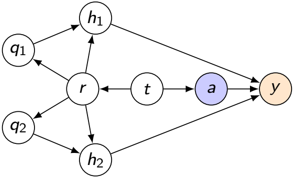

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Problem: Size and variance are at odds!

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Domination Among Adjustment Sets

- Let $Z_1, Z_2 \subseteq V(G)$ be two valid adjustment sets for estimating the causal effect of X on Y in G.
- We write $Z_1 \leq^{\sigma}_{G} Z_2$ if $\sigma^2_{Z_1}(P) \leq \sigma^2_{Z_2}(P)$ for every joint probability dist. P that factorizes according to G.
  - Induces a partial order. There may exist two adjustment sets S_1, S_2 where $\sigma^2_{S_1}(P) < \sigma^2_{S_2}(P)$ and $\sigma^2_{S_2}(P') < \sigma^2_{S_1}(P')$ where P, P' both factorize according to G.
- We say that Z_1 **dominates** Z_2 if:

$$Z_1 \leq^{\sigma}_{G} Z_2 \text{ and } |Z_1| \leq |Z_2|$$

and one of these inequalities is proper.

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Goal: Return all non-dominated adjustment sets.

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Refined Goal: Return all non-dominated adjustment sets whose size (weight) is at most k.

Theorem

There is an algorithm that lists the Pareto-Optimal minimal adjustment sets for computing an unbiased estimator of the interventional mean of outcomes Y under interventions on X, of size at most k, in total time $O(4^k \cdot k \cdot (n+m))$.

Generating the Pareto-Optimal Frontier of Adjustment Sets

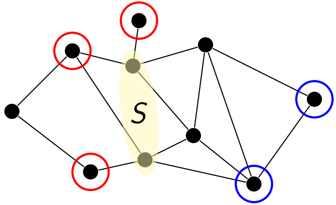
Fixed setting: DAG G; obtain unbiased estimator of the interventional mean of outcomes Y under interventions on X.

Ingredients:

1. Translate problem of finding adjustment sets in DAG G to finding *separators* in an undirected graph $\mathcal{H}$.
   - Validity constraints - we want only adjustment sets $Z \subseteq R$.
   - Inclusion constraints - we want only adjustment sets $I \subseteq Z$.
2. Translate critierion for (partially) ordering adjustment sets based on the variance they yield ($Z_1 \leq_G^{\sigma} Z_2$) to a graphical criterion of separators.

Ingredient #1: From Adjustment Sets to Separators

- Let $\mathcal{H}$ be an undirected graph, and $X, Y \subseteq V(\mathcal{H})$.
- $S \subseteq V(\mathcal{H})$ is an X, Y-separator of $\mathcal{H}$ if in the graph $\mathcal{H} - S$ there is no path from X to Y.
- S is a *minimal* X, Y-separator if none of its proper subsets are.
- S is a *minimum* X, Y-separator of $\mathcal{H}$ if $|S| \leq |S'|$ for every X, Y-separator S'.



Graph $\mathcal{H}$, vertex sets X, $Y \subseteq V(\mathcal{H})$.

We denote by $\mathcal{S}_{X,Y}(\mathcal{H})$ the minimal X, Y-separators in $\mathcal{H}$.

Ingredient #1: From Adjustment Sets to Separators

We denote by $\mathcal{A}_{X,Y}(I, R, G)$ all subsets $I \subseteq Z \subseteq R$ that are adjustment sets for computing an unbiased estimator of the interventional mean of outcomes Y under interventions X. We denote by $\mathcal{A}^{\mathrm{MIN}}_{X,Y}(I, R, G)$ the minimal adjustment sets.

Theorem (based on van der Zander et al. 2019, Smucler et al. 2021)

There is an undirected graph $\mathcal{H}$ derived from G where :

1. $\mathcal{A}_{X,Y}(I, R, G) \neq \emptyset$ *if and only if* X *and* Y *are non-adjacent in* $\mathcal{H}$.
2. $S \in \mathcal{A}^{\mathrm{MIN}}_{X,Y}(I, R, G)$ *if and only if* $S \in \mathcal{S}_{X,Y}(\mathcal{H})$.

Ingredient #2: Graphical Criterion for $\leq_G^\sigma$

- Smucler et al. 2021 presented a graphical criterion for the univariate case - single treatment variable X and single outcome variable Y.
- We generalize to the case where X and Y are variable-sets, and show an equivalent criteria.
- For an X, Y-separator S in $\mathcal{H}$, we denote by:
  - $\mathcal{C}_X(\mathcal{H}-S) \stackrel{\text{def}}{=} \{v \in V(\mathcal{H}) : v \text{ is reachable from } X \text{ in } \mathcal{H}-S\}$.
  - $\mathcal{C}_Y(\mathcal{H}-S) \stackrel{\text{def}}{=} \{v \in V(\mathcal{H}) : v \text{ is reachable from } Y \text{ in } \mathcal{H}-S\}$.

Theorem

Let $S_1, S_2 \in \mathcal{A}_{X,Y}^{\mathrm{MIN}}(I, R, G)$, then:

$$S_1 \leq_G^\sigma S_2 \text{ if and only if } \mathcal{C}_Y(\mathcal{H}-S_1) \subseteq \mathcal{C}_Y(\mathcal{H}-S_2).$$

Graphical Criteria for Domination of Adjustment Sets

Theorem

Let $S_1, S_2 \in \mathcal{A}^{\mathrm{MIN}}_{X,Y}(I, R, G)$*, then:*

$$S_1 \leq^{\sigma}_{G} S_2 \text{ if and only if } \mathcal{C}_Y(\mathcal{H} - S_1) \subseteq \mathcal{C}_Y(\mathcal{H} - S_2).$$

- Causal model G; $R \subseteq V(G)$ are observed; treatment variables X and outcome variables Y.
- Let $\mathcal{H}$ be the undirected graph derived from G where $\mathcal{A}^{\mathrm{MIN}}_{X,Y}(I, R, G) = \mathcal{S}_{X,Y}(\mathcal{H})$ van der Zander et al., Smucler et al. 2019, 2021.
- Let $Z \in \mathcal{S}_{X,Y}(\mathcal{H})$.
- By applying the theorem, we have that $Z \in \mathcal{A}^{\mathrm{MIN}}_{X,Y}(I, R, G)$ belongs to the Pareto frontier if and only if

$$\mathcal{C}_Y(\mathcal{H} - Z') \subseteq \mathcal{C}_Y(\mathcal{H} - Z) \implies |Z| < |Z'|.$$

for every $Z' \in \mathcal{A}_{X,Y}(I, R, G)$.

Pareto Frontier of Adjustment Sets $\equiv$ *Important* Separators

Definition (Important Separator Marx 2011)

Let $S \subseteq V(\mathcal{H})$. We say that S is an important X, Y-separator if $S \in \mathcal{S}_{X,Y}(\mathcal{H})$, and for any other $S' \in \mathcal{S}_{X,Y}(\mathcal{H})$ it holds that:

$$\mathcal{C}_Y(\mathcal{H}-S') \subset \mathcal{C}_Y(\mathcal{H}-S) \implies |S'| > |S|$$

$$\Big\downarrow \text{Ingredients } 1 + 2$$

$S \in \mathcal{A}^{\mathrm{MIN}}_{X,Y}(I, R, G)$ is Pareto optimal $\Leftrightarrow$ $S \in \mathcal{S}_{X,Y}(\mathcal{H})$ and important.

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Theorem (Marx 2011)

There are at most 4^k important X, Y-separators of $\mathcal{H}$ whose size is at most k, and they can all be generated in time $O(4^k \cdot k \cdot (n + m))$.

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$\downarrow$

Theorem

The Pareto-Optimal minimal adjustment sets of size at most k, can be generated in total time $O(4^k \cdot k \cdot (n + m))$.

Is This Enough ?

- This approach works only if k is small.
- Variable weights bounded by constant c.
- Not really ranked: if $k = 10$, then adjustment sets of size 10 may be returned before those of size 3.

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Theorem

Let G be a causal DAG with weight function $w : V(G) \to \mathbb{N}_{\geq 1}$, and $X, Y \subseteq V(G)$ disjoint. There is an algorithm that enumerates all valid adjustment sets for (X, Y) in order of **non-decreasing total weight**, *breaking ties by $\leq_G^\sigma$ (i.e., proximity to Y). The delay is $O(Kn \cdot m^{1+o(1)})$, where K is the size of the largest set listed.*

Conclusion & Outlook

- Our enumeration algorithms generate a comprehensive (pareto-optimal) set of valid, cost-constrained adjustment sets.
- These outputs have the potential to serve as **training data for ML models** that predict high-quality adjustment sets.
- This approach enables data-driven discovery of patterns connecting graphical structure to estimator variance.
  - By analyzing these patterns, we may gain new insights into how the position and composition of adjustment sets affect estimator variance.

Thank you!

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